Sistemi diferencijalnih j-na

$$\begin{cases} x_1' = f_1(t, x_1, \dots, x_n) \\ x_2' = f_2(t, x_1, \dots, x_n) \\ \vdots \\ x_n' = f_n(t, x_1, \dots, x_n) \end{cases}$$ ← normalni sistem DJ (1)

$f_1, \dots, f_n \in C(D)$, $D \subseteq \mathbb{R}^{n+1}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$ $\Rightarrow X' = F(t, X)$ ← matrični oblik (2)

Def. $x_1, x_2, \dots, x_n$ je rešenje sistema (1) ako:

1° $x_i \in \mathcal{D}(G)$, G - interval, $i = \overline{1,n}$

2° $\forall t \in G$ $(t, x_1(t), \dots, x_n(t)) \in D$, D-oblast definisanosti

3° $x_i'(t) = f_i(t, x_1(t), \dots, x_n(t))$ $i = \overline{1,n}$, $\forall t \in G$

Košijev problem za sistem DJ (1): Odrediti rešenje sistema (1) koje zadovoljava uslove:

$$x_1(t_0) = x_1^0, \quad \vdots \quad x_n(t_0) = x_n^0 \qquad (1)(3)$$

-‖- (2): $X_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}$, $X(t_0) = X_0$ (2)(4)

T1: Ako je $F \in C(D)$ $\Rightarrow$ (2)(4) ima rešenje.

T2: Ako je $F \in C(D)$, $(t_0, X_0) \in D$ i $\frac{\partial F}{\partial x_i} \in C(D)$ $\Rightarrow$ (2)(4) ima jedinstveno rešenje

** Linearni sistemi

$$\begin{cases} x_1' = a_{11}x_1(t) + \dots + a_{1n}x_n(t) + b_1(t) \\ \vdots \\ x_n' = a_{n1}x_1(t) + \dots + a_{nn}x_n(t) + b_n(t) \end{cases}$$

$$X' = AX + B$$

Metod eliminacije: (vršimo nad sistemom (1))

$$\left.\begin{aligned} x_1' &= f_1(t, x_1, \dots, x_n) \\ x_1'' &= F_2(t, x_1, \dots, x_n) \\ &\vdots \\ x_1^{(n)} &= F_n(t, x_1, \dots, x_n) \end{aligned}\right\} \xrightarrow{\ \ } \begin{aligned} x_2 &= \Psi_2(t, x_1, x_1', \dots, x_1^{(n-1)}) \\ &\vdots \\ x_n &= \Psi_n(t, x_1, x_1', \dots, x_1^{(n-1)}) \end{aligned}$$

uvrstimo u posledju

$$x_1^{(n)} = F(t, x_1, \dots, x_1^{(n-1)})$$

riješimo x_1

$x_1 = \dots$

1. $x_1' = 2x_1 + x_2$ (1)

$x_2' = 3x_1 + 4x_2$ (2)

R:

(1) $x_1'' = 2x_1' + x_2' = 2(2x_1 + x_2) + 3x_1 + 4x_2 = 7x_1 + 6x_2$

(1) $\Rightarrow \boxed{x_2 = x_1' - 2x_1}$

$x_1'' = 7x_1 + 6(x_1' - 2x_1)$ - linearna DJ sa konstantnim koef.

$x_1'' - 6x_1' + 5x_1 = 0$

$$\begin{cases} x_1 = c_1 e^{t} + c_2 e^{5t} \\ x_2 = x_1' - 2x_1 = \dots \end{cases}$$ opšte rešenje

2. $x_1' = 3x_1 - x_2 + x_3$ (1)

$x_2' = x_1 + x_2 + x_3$

$x_3' = 4x_1 - x_2 + 4x_3$

R $x_1'' = 3x_1' - x_2' + x_3' = 12x_1 \overline{\cancel{x}}\, 5x_2 + 6x_3$ (2)

$x_1''' = 12x_1' + 5x_2' + 6x_3' = 55x_1 - 23x_2 + 31x_3$ (3)

(1) i (2) $\Rightarrow$ $-6\cdot(1) + (2) = -6x_1' + x_1'' = -6x_1 + x_2$

$\boxed{x_2 = x_1'' - 6x_1' + 6x_1}$ (*)

$-5\cdot(1) + (2) \Rightarrow$ $\boxed{x_3 = x_1'' - 5x_1' + 3x_1}$ (**)

uvrstimo (*) i (**) u (3):

$x_1''' - 8x_1'' + 17x_1' - 10x_1 = 0$

$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 5$

$x_1 = c_1e^t + c_2e^{2t} + c_3e^{5t}$

(*) $\Rightarrow x_2 = c_1e^t - 2c_2e^{2t} + c_3e^{5t}$

(**) $\Rightarrow x_3 = -c_1e^t - 3c_2e^{2t} + 3c_3e^{5t}$

3. $x_1' = x_2 + 2e^t$ (1)

$x_2' = x_1 + t^2$ (2)

R $x_1'' = x_2' + 2e^t = x_1 + t^2 + 2e^t$ (*)

(1) $\Rightarrow x_2 = x_1' - 2e^t$

(*) $x_1'' - x_1 = t^2 + 2e^t$ nehomogena sa konstantnim koef.

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$x_h = c_1e^t + c_2e^{-t}$

$\underline{x_{p_1}}$ t^2

$\underline{x_{p_2}}$ $2e^t$

$x_{p_1} = At^2 + Bt + C$

$x_{p_2} = Dte^t$

$$\begin{cases} x_1 = c_1e^t + c_2e^{-t} + (1+t)e^t - 2 - t^2 \\ x_2 = c_1e^t + c_2e^{-t} + te^t - 2t \end{cases}$$

opste reseje

(4.) $x_1' = x_1 - x_2 + 2\sin t$ (1)

$x_2' = 2x_1 - x_2$ (2)

R: $x_1'' = x_1' - x_2' + 2\cos t = x_1 - \cancel{x_2} + 2\sin t - 2x_1 + \cancel{x_2} + 2\cos t$

$x_1'' = -x_1 + 2\sin t + 2\cos t$ (*) nehomogena sa konstantnim koef.

(1) $\Rightarrow x_2 = -x_1' + x_1 + 2\sin t$

(*) : $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$x_h = c_1 \cos t + c_2 \sin t$

x_p $2\sin t + 2\cos t$

$x_p = t(A\cos t + B\sin t)$

$x_p' = A\cos t + B\sin t - At\sin t + Bt\cos t$

$x_p'' = -A\sin t + B\cos t - A\sin t - At\cos t + B\cos t - Bt\sin t =$

$= -2A\sin t + 2B\cos t - At\cos t - Bt\sin t$

$-2A\sin t + 2B\cos t - \cancel{At\cos t} - \cancel{Bt\sin t} + \cancel{At\cos t} + \cancel{Bt\sin t} = 2\sin t + 2\cos t$

$$\left.\begin{aligned} -2A &= 2 \\ 2B &= 2 \end{aligned}\right\} \Rightarrow A = -1,\ B = 1$$

$$\begin{cases} x_1 = c_1\cos t + c_2 \sin t - t\cos t + t\sin t \\ x_2 = x_1 - x_1' + 2\sin t = \ldots \end{cases}$$ opste rešenje

Za vježbu: $x_1' = 5x_1 - 3x_2 + 2e^{3t}$

$x_2' = x_1 + x_2 + 5\cdot e^{-t}$

$x_1 = C_1 e^{2t} + C_2 e^{4t} - e^{-t}$

$$\begin{cases} x_1' = 4x_1 - x_2 \\ x_2' = 3x_1 + x_2 - x_3 \\ x_3' = x_1 + x_3 \end{cases}$$

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V 20.04.2018. (X nedelja)

Homogeni sistem linearnih DJ

$$\begin{cases} x_1' = a_{11} x_1 + \dots + a_{1n} x_n \\ \vdots \\ x_n' = a_{n1}(t) x_1 + \dots + a_{nn} x_n \end{cases}, \quad x_i = x_i(t), \quad a_{ij} \in C(I)$$

u matričnom obliku:

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

$X' = AX$

$L(X) = X' - AX$, $L(X)$ - operator $\Rightarrow L(X+Y) = L(X) + L(Y)$

$L(\lambda X) = \lambda L(X)$

$\Omega = \{ X \in D(I) \mid L(X) = 0 \}$ - prostor rešenja

Lema: Ω je vektorski potprostor.

Lema: $\exists \{X_1, \dots, X_n\}$ baza za Ω.

$$w(t) = \det(X_1, \dots, X_n) = \underbrace{\begin{vmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & & x_{nn} \end{vmatrix}}_{\Phi(t) \text{ fundam. matrica}}(t)$$ determinanta Vronskog

Lema: Sledeći uslovi su ekvivalentni:

1) $\forall t \in I \quad w(t) = 0 \quad (\neq 0)$

2) $\exists t_0 \in I \quad w(t_0) = 0 \quad (\neq 0)$

3) $X_1, \dots, X_n$ su zavisni (nezavisni)

$X = c_1 X_1 + \dots + c_n X_n$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = c_1 \begin{pmatrix} x_{11} \\ \vdots \\ x_{1n} \end{pmatrix} + \dots + c_n \begin{pmatrix} x_{n1} \\ \vdots \\ x_{nn} \end{pmatrix}$$

$\boxed{X = \Phi(t)\cdot C}$, $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

Hom. sis. DJ sa konstantnim koef.

$a_{ij}(t)$ - konstante $\Rightarrow$ (1) hom. sis. DJ sa konst. koef.

$X' = AX$

$X = h\cdot e^{\lambda t}$, h-vektor, $\lambda \in \mathbb{R}$

$\Rightarrow X' = h\cdot e^{\lambda t}\cdot \lambda$

$\lambda h\cdot \cancel{e^{\lambda t}} = \cancel{e^{\lambda t}} A\cdot h$

$\Rightarrow Ah = \lambda h$ $\Rightarrow$ λ mora biti sv. vr, a h odg. sv. vektor

1° $\lambda_1, \dots, \lambda_n$ t.d. $\lambda_i \neq \lambda_j$, $i \neq j$

$\lambda_i \longrightarrow h_i \longrightarrow h_i e^{\lambda_i t}$, $\forall i \in \{1, \dots, n\}$

onda za svaku sv. vr. možemo naći sv. vek.

rešenje je ovog oblika

$X = c_1 h_1 e^{\lambda_1 t} + \dots + c_n h_n e^{\lambda_n t}$ - rešenje

2° λ sv. vr. viš. r

$m = n - \text{rang}(A - \lambda E)$ broj nezavisnih sv. vek.

i) $m = r$

geom. viš. (m), alg. viš. (r)

$\lambda \to h_1 e^{\lambda t}$

$\lambda \to h_2 e^{\lambda t}$

$\vdots$

$\lambda \to h_r e^{\lambda t}$

ii) $m < r$

$e_1 = h_1 e^{\lambda t}$

$\vdots$

$e_m = h_m e^{\lambda t}$

$k = r - m$ pa ostala reš. tražimo u obliku:

$e_{m+1} = (p_0 + \dots + p_k t^k) e^{\lambda t}$

dopunimo nekim polinomom k-tog reda

$\vdots$

$e_r = (q_0 + \dots + q_k t^k) e^{\lambda t}$

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Matrični metod

$X' = AX$

$X = \Phi(t) \cdot C$

$X = e^{At}$?

$x' = ax$

$\frac{dx}{x} = a$

$x = e^{at}$

$$e^{At} = E + At + \frac{A^2}{2!}t^2 + \dots + \frac{A^n}{n!}t^n + \dots$$

pr. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad A^2 = \cancel{\begin{pmatrix} 0 & -1 \\ 1 & 5 \end{pmatrix}}$ $\quad A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$e^{At} = \dots = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$A^4 = E$, $A^5 = A$, $A^6 = A^2, \dots$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(e^{At})' = A \cdot e^{At}$

$X = TY$ (preko matrice prelaska)

pr. $A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k \end{pmatrix}$, $e^{At} = ?$

$$A^n = \begin{pmatrix} a_1^n & & 0 \\ & \ddots & \\ 0 & & a_k^n \end{pmatrix}$$

$$e^{At} = E + tA + \frac{t^2}{2}A^2 + \dots$$

$$(e^{At})_{11} = 1 + ta_1 + \frac{t^2 a_1^2}{2!} + \dots + \frac{t^n a_1^n}{n!} + \dots$$

$$e^{At} = \begin{pmatrix} e^{ta_1} & & 0 \\ & \ddots & \\ 0 & & e^{ta_k} \end{pmatrix}$$

$X = T \cdot Y$ $\quad T$ - konstanta

$X' = TY'$

$X' = AX$

$TY' = ATY \Rightarrow Y' = \underbrace{T^{-1}AT}_{\text{Jordanova matr.}} Y$

$$Y' = JY$$

$$Y = e^{Jt} \cdot C$$

$$\underline{X} = TY = \underline{T \cdot e^{Jt} \cdot C}$$

$$J(\lambda) = \begin{pmatrix} \lambda^1 & & \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}$$

$$e^{J(\lambda)t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \boxed{\frac{t^{r-1}}{(r-1)!}e^{\lambda t}} \\ & \ddots & & \vdots \\ & & & te^{\lambda t} \\ 0 & & & e^{\lambda t} \end{pmatrix}$$

, r-višestrukost korjena polinoma

Zadaci:

(1.) $x_1' = 2x_1 + x_2$

$x_2' = 3x_1 + 4x_2$

R:

Prvo treba odrediti matricu sistema:

$$X' = AX \qquad A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$$8 - 6\lambda + \lambda^2 - 3 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

$\lambda_1 = 1$

$A h_1 = 1 \cdot h_1$

$(A - E) h_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$h_{11} + h_{12} = 0$

$$h_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = 5$

$A h_2 = 5 h_2$

$(A - 5E) h_2 = 0$

$$\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-3h_{21} + h_{22} = 0$

$h_{22} = 3h_{21}$

$$h_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

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$$\Rightarrow X = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{5t} = \begin{pmatrix}e^{t} & e^{5t}\\ -e^{t} & 3e^{5t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$x_1 = c_1 e^{t} + c_2 e^{5t}$$

$$x_2 = -c_1 e^{t} + 3c_2 e^{5t}$$

II način:

$1 \longrightarrow \begin{pmatrix}1\\-1\end{pmatrix}$ s.vr., sv.vek.

$5 \longrightarrow \begin{pmatrix}1\\3\end{pmatrix}$

$$T = \begin{pmatrix}1 & 1\\-1 & 3\end{pmatrix}$$ - matrica promjene baze

$$T^{-1}AT = J = \begin{pmatrix}1 & 0\\0 & 5\end{pmatrix}$$

$$(Y) = e^{Jt} = \begin{pmatrix}e^{t} & 0\\0 & e^{5t}\end{pmatrix}$$

$$Y = e^{Jt}\cdot c$$

$$X = TY = T\cdot e^{Jt}\cdot c =$$

$$= \begin{pmatrix}1 & 1\\-1 & 3\end{pmatrix}\begin{pmatrix}e^{t} & 0\\0 & e^{5t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} =$$

$$= \begin{pmatrix}e^{t} & e^{5t}\\-e^{t} & 3e^{5t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$ - rješenje

2. $x_1' = x_1 - x_2$

$x_2' = x_2 - 4x_1$

$$A = \begin{pmatrix}1 & -1\\-4 & 1\end{pmatrix}$$

$\lambda_1 = 3, \ \lambda_2 = -1$

$\downarrow \qquad \downarrow$

$\begin{pmatrix}1\\-2\end{pmatrix} \qquad \begin{pmatrix}1\\2\end{pmatrix}$ sv. vek.

$$X = c_1\begin{pmatrix}1\\-2\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e^{-t}$$

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II način:

$$T = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^{Jt} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$X = Te^{Jt} \cdot C = \dots$$

3.
$$x_1' = 3x_1 - x_2 + x_3$$
$$x_2' = x_1 + x_2 + x_3$$
$$x_3' = 4x_1 - x_2 + 4x_3$$

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & -1 & 4 \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 1 & 1-\lambda & 1 \\ 4 & -1 & 4-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left((1-\lambda)(4-\lambda)+1\right) + 1(4-\lambda-4) + 1(-1-4+4\lambda) =$$
$$= (3-\lambda)(\lambda^2 - 5\lambda + 5) - \lambda - 5 + 4\lambda =$$
$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 10 = -(\lambda^3 - 8\lambda^2 + 17\lambda - 10) =$$
$$= -(\lambda-1)(\lambda^2 - 7\lambda + 10) = -(\lambda-1)(\lambda-2)(\lambda-5)$$

$\lambda_1 = 1$ $\qquad (A-E)h_1 = 0$

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2h_{11} - h_{12} + h_{13} = 0 \qquad (1)$$

$$h_{11} + h_{13} = 0 \Rightarrow h_{11} = -h_{13} \qquad (2)$$

$$(1) \; -2h_{13} - h_{12} + h_{13} = 0 \Rightarrow h_{12} = -h_{13}$$

$$\Rightarrow h_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & 2 \end{pmatrix}\begin{pmatrix} h_{21} \\ h_{22} \\ h_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (A-2E)h_2 = 0.$$

$h_{21} - h_{22} + h_{23} = 0$

$4h_{21} - h_{22} + 2h_{23} = 0$ $\quad \ominus$

$3h_{21} + h_{23} = 0$

$h_{23} = -3h_{21}$

$h_{22} = h_{21} + h_{23} = h_{21} - 3h_{21} = -2h_{21}$

$$h_2 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

$\lambda_3 = 5$

$$\begin{pmatrix} -2 & -1 & 1 \\ 1 & -4 & 1 \\ 4 & -1 & -1 \end{pmatrix}\begin{pmatrix} h_{31} \\ h_{32} \\ h_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$h_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$X = c_1 h_1 e^{t} + c_2 h_2 e^{2t} + c_3 h_3 e^{5t}$$

$\underline{\text{II}}$ način

$$\lambda_1 = 1 \longrightarrow \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2 \longrightarrow \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

$$\lambda_3 = 5 \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$T = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -2 & 1 \\ 1 & -3 & 3 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$e^{Jt} = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$

$X = T \cdot e^{Jt} \cdot C = \dots$ $\qquad Y = e^{Jt} \cdot C$

$X' = AX$ $\qquad X = T \cdot e^{Jt} \cdot C$

$X = TY$

$X' = TY'$

$TY' = ATY$

$Y' = \underbrace{T^{-1}AT}_{J}\, Y$

4. $x_1' = 3x_1 - x_2$

$x_2' = 4x_1 - x_2$

F

$X' = AX$

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$-3 - 2\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \quad \Rightarrow \lambda = 1,\ r = 2$

$A - 1E = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \longrightarrow$ rang 1

$m = n - \text{rang}(A - \lambda E) = 2 - 1 = 1$ - br. nez. vek.

$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad 2h_{11} - h_{12} = 0$

$h_{12} = 2h_{11}$

$h_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$k = r - m = 2 - 1 = 1$

$\varphi_2 = (p_0 + p_1 t)e^t$, $p_0 = \begin{pmatrix} p_{01} \\ p_{02} \end{pmatrix}$, $p_1 = \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix}$

$\varphi_2' = (p_0 + p_1 t)e^t + p_1 e^t =$

$= (p_0 + p_1 + p_1 t)e^t$

• p_0
• p_1

$\varphi_2' = A\varphi_2$

$A\varphi_2 = (Ap_0)e^t + (Ap_1)t \cdot e^t$

$(p_0 + p_1 + p_1 t)\cancel{e^t} = (Ap_0 + Ap_1 t)\cancel{e^t}$

$\begin{cases} p_1 = Ap_1 & \to p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ p_0 + p_1 = Ap_0 & \iff (A-E)p_0 = p_1 \end{cases}$

$\begin{pmatrix} p_{01} \\ p_{02} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} p_{01} \\ p_{02} \end{pmatrix} = \begin{pmatrix} 3p_{01} - p_{02} \\ 4p_{01} - p_{02} \end{pmatrix}$

$\Rightarrow p_{01} + 1 = 3p_{01} - p_{02} \Rightarrow 2p_{01} - p_{02} = 1$ $\Rightarrow p_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$p_{02} + 2 = 4p_{01} - p_{02} \Rightarrow 4p_{01} - 2p_{02} = 2$

$\varphi_2 = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t\right) \cdot e^t$

$\varphi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t$ (za λ_1)

$X = c_1\varphi_1 + c_2\varphi_2 = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^t + c_2\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t\right)e^t$

$X_1 = c_1 e^t + c_2 e^t + c_2 t e^t$

$X_2 = 2c_1 e^t + c_2 e^t + 2c_2 t e^t$

II način

$1 \to \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, rang $(A - \lambda E) = 1$

$m = 2 - 1 = 1$

$$(A-\lambda E)\,h_2 = h_1$$

$$(A-E)\,h_2 = h_1$$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2h_{21} - h_{22} = 1$$

$$h_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$e^{Jt} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

$$X = T\cdot e^{Jt}\cdot c =$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \boxed{\begin{matrix} e^t \\ 2e^t \end{matrix}} & \boxed{\begin{matrix} -te^t + e^t \\ 2te^t + e^t \end{matrix}} \end{pmatrix}\cdot c$$

ℓ_1 $\quad\quad$ ℓ_2 (iz I načina)

za vježbu: $x_1' = -3x_1 + 2x_2$

$x_2' = -2x_1 + x_2$